Second step algorithms in the Burrows–Wheeler compression algorithm

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November 22, 2001

This is a preprint of an article published in
Software—Practice and Experience, 2002; 32(2):99–111

http://www.interscience.wiley.com

Abstract

In this paper we fix our attention on the second step algorithms of the Burrows–Wheeler compression algorithm, which in the original version is the Move To Front transform. We discuss many of its replacements presented so far, and compare compression results obtained using them. Then we propose a new algorithm that yields a better compression ratio than the previous ones.

KEY WORDS: data compression; lossless compression; Burrows-Wheeler transform; block-sorting; List Update Problem; Calgary Corpus

INTRODUCTION

In 1994, Burrows and Wheeler developed a new data compression algorithm [1]. It is based on the Burrows–Wheeler transform introduced in the same paper. High compression efficiency of this algorithm stimulated a lof of research. Many of its modifications presented so far are described and discussed in Reference [2].

In the first part of this paper we describe the Burrows–Wheeler compression algorithm (BWCA). Later we fix our attention on its second step (Move To Front in the original paper [1]). We discuss recent suggestions of changing it and propose a new algorithm. Next we compare the efficiency of different second step algorithms. Finally we compare the new compression algorithm with other known methods.

THE BURROWS–WHEELER COMPRESSION ALGORITHM

In order to describe the Burrows–Wheeler compression algorithm, we have to introduce some definitions and notations. Let $x = x_1 x_2 \dots x_n$ be a *sequence*. The *length* of the sequence — denoted by n — is the number of elements in the sequence x. Every element x_i of the sequence belongs to a finite ordered set $\mathcal{A} = \{a_0, a_1, \dots, a_{k-1}\}$ which we call an *alphabet*. The *size* of the alphabet, k, is the number of elements belonging to it. We call the elements of the alphabet *symbols* or *characters*. If $x = uvw$ for some possibly

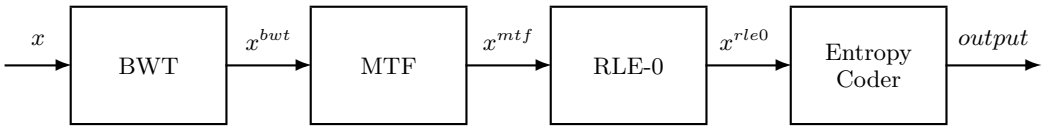

Figure 1: The Burrows–Wheeler compression algorithm.

empty sequences u, v and w, then u is a *prefix*, v a *component*, and w a *suffix* of x. If a component is non-empty and consists of identical symbols then we call it a *run*.

A finite memory Context Tree source [3] ω is defined by a set $\mathcal{S}$ of contexts s (sequences over the alphabet $\mathcal{A}$) of length not greater than the maximum order D ($|s| \leq D$). The set S of contexts should be complete, which means that for every possible sequence x there exists a context $s \in \mathcal{S}$, being a suffix of x. There is exactly one such context, called a proper context. A conditional probability distribution $\{\Theta_s, s \in \mathcal{S}\} = \{\{\theta(a|s), a \in \mathcal{A}\}, s \in \mathcal{S}\}$ is related with each context $s \in \mathcal{S}$.

The algorithm

The scheme of the Burrows–Wheeler compression algorithm is presented in Figure 1. It is composed of four stages. The first step performs the Burrows–Wheeler transform (BWT). The second step implements the Move To Front (MTF) transform or other algorithm converting the output of the BWT (sequence x^{bwt}) to a form that can be better compressed by an entropy coder. The third step is a specialized version of a Run Length Encoding (RLE) called RLE-0 (in some versions of the BWCA this step does not occur). The last step is an entropy coder, which typically is an arithmetic coder. A more precise description of these algorithms can be found in Reference [2].

For completeness, we should notice that some early versions of BWCA used another kind of RLE before the BWT step to speed up the sorting procedure. Today, however, fast ways to compute BWT are known and it is not necessary to apply this preliminary step. For this reason we will not consider this kind RLE further in the paper.

An example of the Burrows–Wheeler transform

Now we show an example of the operation of BWT that will be helpful for further discussion. Let $x =$ *abracadabra*\$. The last character of the sequence x, \$, is called a *sentinel*. It is the last character in the alphabet and it appears exactly once in the sequence x. In fact, the sentinel is not part of the data that we want to compress, and is appended to the sequence before the BWT stage.

At the beginning of the BWT, we form a matrix $M(x)$ of cyclic shifts of x:

$$M(x) = \begin{bmatrix} a & b & r & a & c & a & d & a & b & r & a & \$ \\ b & r & a & c & a & d & a & b & r & a & \$ & a \\ r & a & c & a & d & a & b & r & a & \$ & a & b \\ a & c & a & d & a & b & r & a & \$ & a & b & r \\ c & a & d & a & b & r & a & \$ & a & b & r & a \\ a & d & a & b & r & a & \$ & a & b & r & a & c \\ d & a & b & r & a & \$ & a & b & r & a & c & a \\ a & b & r & a & \$ & a & b & r & a & c & a & d \\ b & r & a & \$ & a & b & r & a & c & a & d & a \\ r & a & \$ & a & b & r & a & c & a & d & a & b \\ a & \$ & a & b & r & a & c & a & d & a & b & r \\ \$ & a & b & r & a & c & a & d & a & b & r & a \end{bmatrix}.$$

Then we sort the rows of $M(x)$ in lexicographic order, obtaining the matrix $\widetilde{M}(x)$:

$$\widetilde{M}(x) = \begin{bmatrix} \underline{a} & \underline{b} & \underline{r} & \underline{a} & \underline{c} & \underline{a} & \underline{d} & \underline{a} & \underline{b} & \underline{r} & \underline{a} & \underline{\$} \\ a & b & r & a & \$ & a & b & r & a & c & a & \mathbf{d} \\ a & c & a & d & a & b & r & a & \$ & a & b & \mathbf{r} \\ a & d & a & b & r & a & \$ & a & b & r & a & \mathbf{c} \\ a & \$ & a & b & r & a & c & a & d & a & b & \mathbf{r} \\ b & r & a & c & a & d & a & b & r & a & \$ & \mathbf{a} \\ b & r & a & \$ & a & b & r & a & c & a & d & \mathbf{a} \\ c & a & d & a & b & r & a & \$ & a & b & r & \mathbf{a} \\ d & a & b & r & a & \$ & a & b & r & a & c & \mathbf{a} \\ r & a & c & a & d & a & b & r & a & \$ & a & \mathbf{b} \\ r & a & \$ & a & b & r & a & c & a & d & a & \mathbf{b} \\ \$ & a & b & r & a & c & a & d & a & b & r & \mathbf{a} \end{bmatrix}.$$

The result of BWT is a sequence x^{bwt} that appears in the last column of the matrix $\widetilde{M}(x)$, plus the index, $R(x)$, of the row of $\widetilde{M}(x)$ that contains the original sequence x. In our example, we get $x^{bwt} = \$drcraaaabba$ and $R(x) = 1$.

The sentinel is emloyed because sorting the cyclic shifts is a computationaly intensive problem. For a sequence ended by a sentinel it reduces to the problem of sorting suffixes. The latter problem can be solved by building a suffix tree and searching it in lexicographic order, which can be done very effectively. The other advantage of using the sentinel is better relation of BWT to Context Tree sources.

BWT relation to Context Tree sources

Balkenhol and Kurtz [4] showed that if we assume that the sequence x is generated by a Context Tree source, then the BWT groups together the similar contexts. This means that the symbols appearing in a specified context $s \in \mathcal{S}$ are grouped together in the last column of the consecutive rows of the matrix $\widetilde{M}(x)$. Futhermore the probability distribution in a component of the sequence x^{bwt}, related to one context s (CT-component), does not change. The contexts contained in the set $\mathcal{S}$ appear in $\widetilde{M}(x)$ in the lexicographic order, so the sequence x^{bwt} is a concatenation of the CT-components, each of them corresponding to one of the leaves in the Context Tree source model.

PREVIOUS WORKS ON THE SECOND STEP ALGORITHMS

As was mentioned above, the sequence x^{bwt} is a concatenation of components corresponding to separate contexts. Unfortunately, there is no information where exactly each such a CT-component starts in the sequence x^{bwt}. (This is the main difference between the BWCA and the PPM algorithms.) However, monitoring the probability distribution in the sequence x^{bwt} we can try to uncover some of this information [5, 6]. Surely the sequence x^{bwt} is a permutation of x. To exploit the properties of the sequence x^{bwt}, we have to transform its local structure into a global one in some way. Alternatively, we have to find a way to rapidly adapt to a changing probability distribution in the sequence x^{bwt} without information where the CT-components starts.

Several algorithms to solve this problem have been proposed. Some of them are based on the observation that the problem is similar to the List Update Problem (LUP) [7]. (The review of many algorithms solving the LUP was recently presented by Bachrach and El-Yaniv [8].)

The formulation of the LUP states that there is a list of items and a sequence of requests. A request can be an insertion, a deletion or an access to an item in the list. The algorithm solving the problem must serve these requests. The cost of serving an access request to an item p on the ith position from the front of the list is i, which is the number of comparisons needed to find p. After processing a request, the algorithm can reorganize the list in order to minimize the total cost of its maintainance. Once an item p is accessed, it may be moved free of charge to any position closer to the front of the list (*free transpositions*). Other transpositions, with elements located closer than p to the end of the list are called *paid* and cost 1. (In the BWCA, we have a list of items L, containing the characters from alphabet $\mathcal{A}$, and a sequence of requests x^{bwt}. The list L contains all the possible items, so we never do any insertions or deletions.) The algorithms solving the LUP should minimize the total cost of maintaing the list (the sum of all costs of serving the items from the sequence of requests).

In the second step of the BWCA we do not want to minimize the total cost of maitaining the list L. However, when we apply the algorithm solving the LUP to the list L and the sequence of requests x^{bwt}, we obtain the sequence x^{lup} that contains the numbers corresponding to the costs of accesses the symbols from x^{bwt}. When we sum all the numbers from the sequence x^{lup}, we get the total cost of maintaing the list L (we assume here that the algorithm for the LUP does not perform paid transpositions). The main goal of the second step of the BWCA is not to minimize the total cost, however typically when the total cost is smaller, then the distribution of probabilities of symbol occurences in the sequence x^{lup} is less uniform. In the last step of the BWCA, we apply the entropy coder to the sequence x^{lup}, and when the distribution of the symbols in this sequence is less uniform then a better compression ratio can be achieved. This observation is of course not precise, because the compression ratio depends on the probability distribution of symbols in the sequence x^{lup} in a more complicated way. When we use the RLE-0 algorithm, we in fact encode the sequence x^{rle0}, whose probability distribution is slightly different. The overall compression ratio depends also on the probability estimation method in the last step of the BWCA. However, typically minimizing the total cost of maintaining the list L yields a better compression ratio.

We also note that the additional cost of paid transpositions can be neglected, because the symbols in x^{lup} correspond to the cost of finding the current item, and there is no cost related to reorganizing the list L. In particular, we can say that in the BWCA the problem is similar to the modified List Update Problem, in which the paid transpositions cost 0. The more so, the sequence x^{bwt} has a specific structure and there are many dependencies in it. Therefore using the best algorithms specialized for the LUP does not always lead to the best compression results. Several modifications to these algorithms were proposed to exploit the properties of the BWT in a better way.

Move To Front and its modifications

Burrows and Wheeler [1] suggested using the Move To Front transform (MTF) [9] as the second step of the BWCA. The MTF maintains a character list L. When a character x_i^{bwt} appears, the list L is scanned and the position of the character x_i^{bwt} in the list L is assigned to x_i^{mtf}. (As result we get the sequence x^{mtf} over the alphabet $\mathcal{A} = \{0, 1, \ldots, k-1\}$.) Then the character is moved to the beginning of the list L. This is a very simple strategy, but it gives quite good results is practice.

Burrows and Wheeler suggested that refraining from moving the current character to the very first position may be sometimes useful. Fenwick [10, 11] and Schindler [12] explored such a possibility but failed to obtain better compression results. Recently Balkenhol, Kurtz and Shtarkov [13] proposed a modification called MTF-1 which improves the compression ratio. Their only modification to the MTF algorithm is that only the symbols from the second position in the list L are moved to the first position. The symbols from higher positions are moved to the second position. Balkenhol and Shtarkov [6] proposed a further more modification to the MTF-1—the symbols from the second position are moved to the begining of the list L only when the previous transformed symbol was at the first position (following the authors we call this version MTF-2).

Time Stamp

One of the best algorithms for the List Update Problem is Time Stamp presented by Albers [14]. The deterministic version of this algorithm—TimeStamp(0) (TS(0)) [15]—scans for a processed character x_i^{bwt} in the list L and outputs its position. Then it moves the character in front of the first item in the list L that has been requested at most once since the last request to the character x_i^{bwt}. If the character x_i^{bwt} has not been requested so far, it is left at the same position.

TS(0) was theoretically analysed by Albers and Mitzenmacher [15]. They show that theoretically the TS(0) is better than then MTF. The results obtained for sequences from Calgary Corpus [16] are worse than with the MTF, but the authors do not provide them explicitly.

Inversion Frequencies

The completely new approach to the problem of transforming the sequence x^{bwt} to a form that can be better compressed by an entropy coder was proposed by Arnavut and Magliveras [17]. They described an algorithm called Inversion Frequencies (IF). The IF algorithm is not the algorithm solving the LUP. It forms a sequence x^{if} over an alphabet of integers from the range $[0, n-1]$. For each character a_j from the alphabet, the IF algorithm scans the sequence x^{bwt}. When it finds the first occurence of the character a_j it outputs its position in the sequence x^{bwt}. For further occurences of the character a_j the IF outputs an integer which is the number of characters greater than a_j that occurred since the last request to the character a_j. The sequence x^{if} is not sufficient to recover the sequence x^{bwt} correctly. We also have to know the number of occurrences of each character from the alphabet in the sequence x^{bwt}. This disadvantage is especially important for short sequences.

Distance Coding

Recently a new proposition for the second step algorithm—Distance Coding (DC)—was suggested by Binder. This algorithm has not been published yet, and we describe it following References [18, 19].

For each character x_i^{bwt}, DC finds its next occurence in the sequence x^{bwt} (x_p^{bwt}) and outputs the distance to it, i.e. $p - i$. When there are no further occurrences of x_i^{bwt}, DC outputs 0. To estabilish the sequence

x^{bwt} correctly, we also have to know the first position of all the alphabet characters. The basic version of DC, described above, is in fact a small modification of Interval Encoding proposed by Elias [20].

To improve the compression ratio, Binder proposed three improvements to the basic algorithm. First, we can notice that in the sequence x^{dc} some of the ending zeroes are redundant. Second, when we are scanning the sequence x^{bwt} for the next occurence of the current character, we may count only the characters that are unknown at this moment. Third—the most important—if the next character is the same as the current one, we do not need to encode anything, and we can simply proceed to the next character.

The Balkenhol–Shtarkov method for coding MTF output

Recently Balkenhol and Shtarkov [6] proposed a new approach to the coding of the sequence x^{bwt}. Their method is based on the observation that the entropy coder codes the sequence over the alphabet $\mathcal{A}^{mtf}$ (or $\mathcal{A}^{rle0}$) consisting of integers. For typical sequences x, the probability distribution of the integers in x^{mtf} decreases monotonicaly for larger integers. Unfortunately this distribution varies in the sequence and—what is worse— for a given integer, we do not know which character it represents. For example, two identical integers greater then 1 that appear at succesive positions in x^{mtf} represent different characters.

Balkenhol and Shtarkov suggest dividing the sequence x^{mtf} into two sequences[1]. The first sequence, $x^{mtf,1}$, is over the alphabet $\mathcal{A}^{mtf,1} = \{0, 1, 2\}$. It is constructed from x^{mtf} by replacing all ocurrences of integers greater than 1 with 2. This means that the ternary sequence $x^{mtf,1}$ holds only the information whether the transformed character was at the first (integer 0), the second (integer 1) or at some other position (integer 2) in the list L. The second sequence, $x^{mtf,2}$, is over the alphabet $\mathcal{A}$. It is constructed from the sequence x^{bwt} by removing the characters for which the integers 0 or 1 appear in the sequence $x^{mtf,1}$.

Further, the sequence $x^{mtf,1}$ is treated as a Markov chain and is encoded using the standard universal coding scheme. To encode the sequence $x^{mtf,2}$ the alphabet is divided into four groups of changing sizes and contents. The symbols from the sequence $x^{mtf,2}$ are encoded with the probability estimation from one of these groups. The groups are maintained in such a way that they work as sliding windows and contain statictics of symbol occurrences in these windows. Because with the high probability the last symbols have similar probability distribution, such an estimation helps to obtain good compression results.

WEIGHTED FREQUENCY COUNT

We propose a new algorithm, which can be considered as a generalization of the well known Frequency Count (FC). We call this algorithm a Weighted Frequency Count (WFC).

First we formulate the FC algorithm in an alternative way. We assign to each character a_j appearing prior to the ith position in the sequence x^{bwt} a sum

$$W_i(a_j) = \sum_{\substack{1 \le p < i \\ a_j = x_p}} 1$$

and sort the list L according to the decreasing values of counters $W_i(a_j)$.

Next, we note that instead of summing 1's for all characters we can conditioned the numbers that are summed from its relative position in the sequence x^{bwt}. To this end we introduce the weight function w and reformulate the sum

[1] In fact Balkenhol and Shtarkov use the MTF-2 algorithm instead of the MTF and the sequence x^{mtf-2}.

$$W_i(a_j) = \sum_{\substack{1 \le p < i \\ a_j = x_p}} w(i-p).$$

If two characters have the same value $W_i(.)$, we find their relative order using the values $W_{i-1}(.)$, $W_{i-2}(.)$ and so on, until the counters are different. For completion we define $W_0(a_j) = -j$. The algorithm outputting the position of processed characters in the list L and maintaing the list in the described way we call the Weighted Frequency Count.

Let us notice that if we set $w(t) = 1, \text{ for } t > 0$, we obtain the FC algorithm and if we set

$$w(t) = \begin{cases} 1, & \text{for } t = 1, \\ 0, & \text{for } t > 1, \end{cases}$$

we obtain the MTF algorithm.

The algorithm Sort By Time (SBT)—recently proposed by Schulz [21]—is also a special case of the WFC algorithm. We get it by setting $w(t) = q^t, \text{ for } t > 0$. Theoretical properties of the SBT algorithm for different values of q has been examined by Schulz. He has shown that by estabilishing $0 < q \le 0.5$ we obtain the MTF algorithm.

WFC relation to Context Tree sources

As mentioned above, the sequence x^{bwt} is a concatenation of the CT-components. Therefore the character which has occurred at the previous position is more likely to be at the same context than the last but one, and so on. Generally, the characters which have appeared at recent positions are more likely described by the same probability distribution as the current character than the ones from more distant positions. Unfortunately, we do not know where in the context tree we are, how long the CT-component of the current leaf is, and where in that CT-component we are.

There is also one more property: typically similar contexts have only slightly different probability distribution (this property is one of the basis of the PPM algorithms). It may be useful to explore some of the information of probability distribution of previous contexts. All in all, the weight function w should decrease. It is not clear, however, how fast it should decrease.

We have explored many functions w. Our results show that for different sequences from Calgary Corpus different functions w give best results. This is of course what one would expect. Short sequences have typically shorter CT-components than longer ones. Also the sequences x are generated by different sources which may have different number of contexts.

An efficient implementation

The formulation of the WFC algorithm does not give us a method for fast computing the values of $W_i(.)$. When we switch to the next character, we have to recalculate the values of all counters $W_i(.)$. To this end, we have to rescan the encoded part of the sequence x^{bwt}. This yields the computational complexity $O(n(n + k \log k))$.

The complexity of the WFC algorithm can be improved by sacrifying its precision. One possibility is to quantize the values of w to integer powers of 2. This decreases the number of different values of w to at most $l = \log_2 w(1)/w(t_{max}) + 1$ (we assume that the values w are decreasing), which is typically small. For such values of w we can obtain the values of $W_i(.)$ from $W_{i-1}(.)$ by updating only the counters for the characters where the values w are changing (for all such t that $w(t) \neq w(t-1)$). Using this approach we obtain the

$$
\begin{aligned}
w_1(t) &= \begin{cases} 1 & \text{, for } t = 1 \\ 0 & \text{, for } t > 1 \end{cases} \\
w_2(t) &= \begin{cases} q^t & \text{, for } 1 \le t \le t_{max} \\ 0 & \text{, for } t > t_{max} \end{cases} \\
w_3(t) &= \begin{cases} \frac{1}{p*t} & \text{, for } 1 \le t \le t_{max} \\ 0 & \text{, for } t > t_{max} \end{cases} \\
w_4(t) &= \begin{cases} 1 & \text{, for } t = 1 \\ \frac{1}{p*t} & \text{, for } 1 < t \le t_{max} \\ 0 & \text{, for } t > t_{max} \end{cases} \\
w_5(t) &= \begin{cases} 1 & \text{, for } t = 1 \\ p * t^q & \text{, for } 1 < t \le t_{max} \\ 0 & \text{, for } t > t_{max} \end{cases} \\
w_6(t) &= \begin{cases} 1 & \text{, for } t = 1 \\ \frac{1}{p*t} & \text{, for } 1 < t \le 64 \\ \frac{1}{2*p*t} & \text{, for } 64 < t \le 256 \\ \frac{1}{4*p*t} & \text{, for } 256 < t \le 1024 \\ \frac{1}{8*p*t} & \text{, for } 1024 < t \le t_{max} \\ 0 & \text{, for } t > t_{max} \end{cases}
\end{aligned}
$$

Figure 2: Examined weight functions.

algorithm of the worst-case complexity $O(nlk)$, which is not much greater than $O(nk)$ for algorithms like the MTF. In practice the characters on the list L move only by a small number of positions. Using a modified insertion sort algorithm the cost of maintaining the list is small (e.g. for the function w_{6q} the average number of swaps of characters on the list L per input character is less than 6 for almost all files from Calgary Corpus, except for binary files such as `geo` where it is larger).

The disanvantage of using this approach in the compression ratio depends on the weight function w and properties of the sequence. If necessary we can double the number of different values of w by using also the powers of 2 for half exponents.

Comparison of different weight functions

In the first experiment we have examined many weight functions w. In this section we describe the compression results obtained for some of them (see Figure 2). In our experiments we have used the BWT-based compression algorithm described in Reference [2] (D99), where the MTF-1 algorithm was replaced by the WFC algorithm with the examined weight functions w.

For each of the examined weight functions we have looked for the best set of parameters. The obtained results are shown in Table 1. We achieve the best overall results for the function w_6. The use of different

Table 1: Compression results for different weight functions (for all functions we used $t_{max} = 2048$).

File	Size	weight functions w_1	w_2 $q = 0.7$	w_3 $p = 4$	w_4 $p = 4$	w_5 $p = 0.5$, $q = -1.25$	w_6 $p = 4$	w_{6q} $p = 4$
bib	111261	1.915	1.916	1.969	1.916	1.899	**1.896**	**1.896**
book1	768771	2.344	2.311	2.280	2.283	2.279	**2.273**	2.274
book2	610856	1.999	1.980	1.999	1.973	1.962	1.959	**1.958**
geo	102400	4.235	4.229	**4.115**	4.121	4.146	4.150	4.152
news	377109	2.464	2.461	2.464	2.415	2.410	**2.409**	**2.409**
obj1	21504	3.766	3.757	3.724	**3.695**	**3.695**	3.697	**3.695**
obj2	246814	2.439	2.448	2.492	2.432	2.416	**2.413**	2.414
paper1	53161	2.420	2.422	2.488	2.424	2.405	**2.403**	**2.403**
paper2	82199	2.382	2.370	2.405	2.364	2.351	**2.347**	**2.347**
pic	513216	0.761	0.741	**0.703**	0.706	0.716	0.718	0.717
progc	39611	2.455	2.461	2.521	2.451	**2.431**	**2.431**	**2.431**
progl	71646	1.684	1.697	1.769	1.682	1.672	**1.670**	**1.670**
progp	49379	**1.667**	1.690	1.787	1.690	1.673	1.672	1.672
trans	93695	**1.450**	1.483	1.611	1.466	1.456	**1.450**	1.452
Avg	3141622	2.284	2.283	2.309	2.258	2.251	2.249	2.249

parameters for different ranges in the function w_6 is motivated by the fact that typically characters in these ranges are from different (but similar) contexts. It is useful to exploit the information of the probability distribution in such contexts, but it should not dominate the probability distribution of the current context.

The last column (w_{6q}) contains the results for the function w_6, where all values were replaced by the nearest negative integer powers of 2. As we can see, the disanvantage caused by the quantization can be neglected. However, for other functions w this difference may not be so small. In further experiments we will use the best of the examined functions in the quantized form—w_{6q}.

COMPARISON OF COMPRESSION ALGORITHMS

In the second experiment, we compare the compression results of different second step algorithms (see Table 2). The results in columns denoted by MTF, MTF-1, MTF-2, TS(0) and WFC are obtained using a compression method described in Reference [2], where the MTF-1 algorithm was replaced by the mentioned ones. For the rest of the algorithms different probability estimation is needed and we cannot simply replace only the second step algorithm. The results for the DC algorithm are taken from experiments with its implementation [23]. The results for the BS99 algorithm are obtained from Reference [6]. The IF algorithm was implemented as follows. For each number x_i^{if} the Elias code [22] of length $1 + \lfloor 2 \log x_i^{if} \rfloor$ bits is calculated and encoded using a binary arithmetic coder. Each bit of the Elias code is encoded in a separate context, where the probability estimation is calculated as weighted probability for orders 0 and 2 (similarly to Reference [2]). Instead of encoding the number of occurences of all characters we have modified the IF algorithm as follows. When we process a character a_j and reach the end of the sequence x^{bwt}, we increment

Table 2: Comparison of different second step algorithms for the files from Calgary Corpus.

File	Size	MTF	MTF-1	MTF-2	TS(0)	IF	DC	BS99	WFC
bib	111261	1.915	1.904	1.904	2.012	1.963	1.931	1.91	**1.896**
book1	768771	2.344	2.317	2.304	2.308	**2.239**	2.241	2.27	2.274
book2	610856	1.999	1.983	1.976	2.027	1.964	**1.938**	1.96	1.958
geo	102400	4.235	4.221	4.220	4.186	4.190	4.510	4.16	**4.152**
news	377109	2.464	2.450	2.449	2.586	2.459	**2.397**	2.42	2.409
obj1	21504	3.766	3.737	3.740	3.900	3.889	3.969	3.73	**3.695**
obj2	246814	2.439	2.427	2.429	2.637	2.548	2.451	2.45	**2.414**
paper1	53161	2.420	2.411	2.411	2.588	2.454	2.407	2.41	**2.403**
paper2	82199	2.382	2.369	2.364	2.458	2.366	**2.343**	2.36	2.347
pic	513216	0.761	0.741	0.737	0.732	**0.706**	0.717	0.72	0.717
progc	39611	2.455	2.446	2.450	2.643	2.500	2.473	2.45	**2.431**
progl	71646	1.684	1.678	1.681	1.851	1.747	1.692	1.68	**1.670**
progp	49379	1.667	**1.665**	1.670	1.887	1.745	1.705	1.68	1.672
trans	93695	1.450	**1.448**	1.452	1.704	1.557	1.473	1.46	1.452
Avg	3141622	2.284	2.271	2.270	2.394	2.309	2.303	2.26	2.249

the current character to the a_{j+1} and count further from the beginning of the sequence x^{bwt}. Therefore we sometimes can get number greater than n, but when we use the binary arithmetic coder and encode each bit separately it is not a problem.

The results show that different algorithms are best for different files from Calgary Corpus, but most of the top results are obtained using the WFC algorithm. However, one should remember that improving the probability estimation for the IF, DC or BS99 algorithms may change this result.

In the third experiment we compare our modified algorithm D00 (D99 when the MTF-1 algorithm is replaced by the WFC algorithm) to the best known BWT-based and other algorithms. Table 3 contains the compression results as output bits per input character (bpc) for following methods:

- gzip—standard *gzip* program with option –9 (this is an implementation of the well known LZ77 algorithm [24]),
- PPMD+—PPMD+ algorithm implemented by Teahan [25],
- B97—the best version of PPM algorithms proposed by Bunton [26],
- CTW—Context Tree Weighting method proposed by Willems, Shtarkov and Tjalkens [27],
- BW94—original Burrows–Wheeler algorithm [1],
- VW98 — switching method, algorithm presented by Volf and Willems [28, 29],
- D99—BWT-based algorithm proposed by Deorowicz [2],
- BS99—BWT-based algorithm proposed by Balkenhol and Shtarkov [6],
- D00—BWT-based algorithm proposed in this paper.

Table 3: Comparison of compression algorithms for Calgary Corpus.

File	Size	gzip	PPMD+	B97	CTW	VW98	BW94	D99	BS99	D00
bib	111261	2.509	1.862	1.786	1.79	1.714	2.07	1.904	1.91	1.896
book1	768771	3.250	2.303	2.184	2.19	2.150	2.49	2.317	2.27	2.274
book2	610856	2.700	1.963	1.862	1.87	1.820	2.13	1.983	1.96	1.958
geo	102400	5.345	4.733	4.458	4.46	4.526	4.45	4.221	4.16	4.152
news	377109	3.063	2.355	2.285	2.29	2.210	2.59	2.450	2.42	2.409
obj1	21504	3.839	3.728	3.678	3.68	3.607	3.98	3.737	3.73	3.695
obj2	246814	2.628	2.378	2.283	2.31	2.245	2.64	2.427	2.45	2.414
paper1	53161	2.791	2.330	2.250	2.25	2.152	2.55	2.411	2.41	2.403
paper2	82199	2.887	2.315	2.213	2.21	2.136	2.51	2.369	2.36	2.347
pic	513216	0.817	0.795	0.781	0.79	0.764	0.83	0.741	0.72	0.717
progc	39611	2.678	2.363	2.291	2.29	2.195	2.58	2.446	2.45	2.431
progl	71646	1.805	1.677	1.545	1.56	1.482	1.80	1.678	1.68	1.670
progp	49379	1.812	1.696	1.531	1.60	1.460	1.79	1.665	1.68	1.672
trans	93695	1.611	1.467	1.325	1.34	1.256	1.57	1.448	1.46	1.452
Avg	3141622	2.695	2.283	2.177	2.19	2.123	2.43	2.271	2.26	2.249

The algorithm D00 yields the best results in the class of BWT-based algorithms, though its advantage over the BS99 is rather small (only about 0.01 bpc).

CONCLUSIONS

We have described different second step algorithms for the Burrows–Wheeler compression algorithm. We have also presented a new algorithm (Weighted Frequency Count) for the second step. The results of our ad hoc proposed functions w are promising. We expect that future experiments could give further improvements in the compression ratio. Also a theoretical analysis of the WFC algorithm should be provided. It would be interesting to find more precise relation of the presented weight functions w to the CT-source model.

The overall results for the best BWT-based compression algorithms are close to the results of the best compression methods known. We expect that further improvements are possible and we can get even closer to the best algorithms. It is however probable that BWT-based algorithms cannot match the performance of the PPM algorithms, because the knowledge of the locations where the contexts change gives an important advantage to the latter family of algorithms.

ACKNOWLEDGEMENTS

The research was carried out under the Silesian University of Technology Research Project Bk-210/RAu2/2000. The author wishes to thank N. Jesper Larsson for the source code of the suffix tree creation algorithm and John Carpinelli, Alistair Moffat, Radford Neal, Wayne Salamonsen, Lang Stuiver and Ian Witten for the source code of the arithmetic coding algorithm which were both used in the experiments. I also wish to thank Zbigniew J. Czech, Marcin Ciura and Roman Starosolski for reviewing the manuscript and suggesting

improvements, to Szymon Grabowski for pointing out the DC algorithm and for interesting discussion on the BWT-based compression algorithms, and to anonymous referees for helpful comments.

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